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Existence and uniqueness of a nonlinear term realizing a prescribed blow-up time

By

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Abstract

This article is a survey of recent papers ([13, 14]) by the authors on an inverse problem of determining a super-linear, nonlinear term of a simple differential equation from information on blow-up time of solutions to the equation. We expand local and global theory to find its nonlinear term realizing a prescribed blow-up time that is a function of an initial data of the differential equation, as well as, to answer the uniqueness question whether the nonlinear term is unique. Some unpublished results, useful examples, and an overview of main ideas for the proof are also contained.

§ 1. Problem

Blow-up phenomena of solutions to nonlinear differential equation arise in many fields of mathematical physics. In order to provide a way to understand their nature, and also, to answer an applied question whether one can determine a nonlinearity (such as a driving force) of a model equation from an observed data-set of blow-up time (such as time to disappear in extraterrestrial), we consider an inverse problem to determine a nonlinear term of a differential equation from information concerning blow-up time of solutions to the equation.

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The model equation in this article is the following simple system:

$$(1.1)\qquad \begin{cases} \dfrac{d^2u}{dt^2} = f(u), & 0 < t < \infty; \\ u(0) = h, & a_0 < h < \infty; \\ \dfrac{du}{dt}(0) = 0. \end{cases}$$

Here $-\infty \leq a_0 < \infty$ and f is a positive, continuous function on the interval (a_0, ∞). When f is super-linear, the solution $u = u(t, h)$ of (1.1) blows up at the (finite) time $T_f(h)$ for each $h \in (a_0, \infty)$ (see Figure 1).

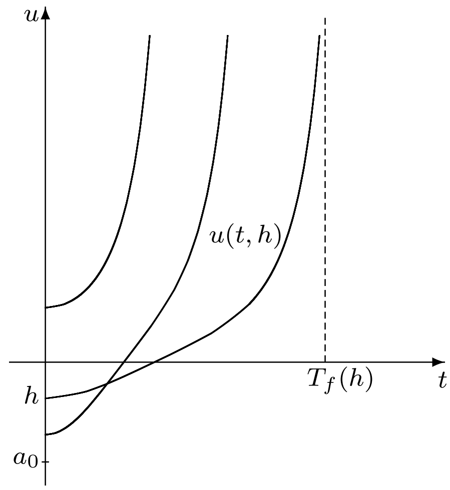

Figure 1. Blow-up of solutions.

Then we get the correspondence

$$\mathcal{B} : f \longmapsto T,$$

which assigns the blow-up time function $T = T_f(h)$ to the nonlinear term f. We call $\mathcal{B}$ the blow-up time map.

We now pose our inverse problem (see Figure 2):

Problem (Inverse blow-up problem) Investigate $\mathcal{B}^{-1}$.

In particular, we are interested in the following issues:

(1) Given T, does there exist f such that $\mathcal{B}f = T$?

(2) Is f unique?

This inverse blow-up problem was posed in [13], motivated by a theoretical use of blowing up solutions to various differential equations for discussions (see [21, 22]) based upon comparison theorems. As was mentioned in [13], the solution $u(t, h)$ of (1.1) is

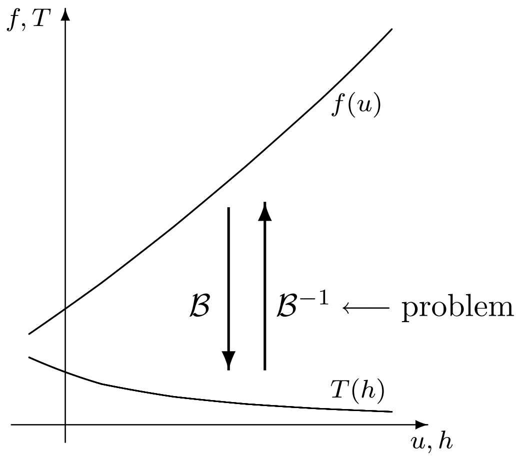

Figure 2. Inverse blow-up problem.

unique with $\frac{du}{dt} > 0$ for each $h \in (a_0, \infty)$ because $\frac{d^2u}{dt^2}$ is positive under the assumption $f > 0$, namely $u(t,h)$ is convex downward, and is given by the inverse function of $t(u)$ defined by

$$\frac{dt}{du} = \frac{1}{\sqrt{2}\sqrt{\int_h^u f(\xi)d\xi}}, \quad t(h) = 0,$$

which is deduced from a standard, conservation formula $\frac{1}{2}\left(\frac{du}{dt}\right)^2 = \int_h^{u(t)} f(\xi)d\xi$. Hence our inverse problem is equivalent to solving the nonlinear integral equation

$$\text{(1.2)} \qquad \frac{1}{\sqrt{2}}\int_h^\infty \frac{du}{\sqrt{\int_h^u f(\xi)d\xi}} = T(h), \quad a_0 < h < \infty,$$

for unknown f with a prescribed function $T(h)$.

Apart from that this equation is nonlinear, it has some features that make its treatment difficult: firstly, it is of the first kind; secondly, it has a singularity at $u = h$; thirdly, it is also singular at $u = \infty$. This article presents not merely results answering two issues relevant to the inverse blow-up problem in Sections 2, 3 but also methods to overcome those difficulties in Section 4, where, in addition, through these methods, we refer to some inverse problems to determine a nonlinear term which are closely related with the inverse blow-up problem.

§ 2. Local theory

In our problem, initial point is $h = +\infty$, because, as is seen from integral equation (1.2), the value of the blow-up time function $T(h)$ at each point h is determined by only the section of f on $[h, \infty)$. Accordingly our first task is to solve this integral equation near $+\infty$, say, on the interval (H, ∞) with sufficiently large H.

§ 2.1. Preliminaries

We employ the following, weighted Hölder space $\mathcal{C}^\alpha(I)_\eta$ of real-valued functions: let I be an interval in $\mathbb{R}$, let $0 < \alpha \le 1$, $\eta \in \mathbb{R}$, and let

$$\mathcal{C}^\alpha(I)_\eta$$
$$:= \left\{ \phi \in C(I) \,:\, ||\phi||_{\alpha,\eta,\,I} := \sup_{x\in I} \big|\, |x|^{-\eta}\phi(x) \big| + \sup_{\substack{x,y\in I\\ x\neq y}} \frac{||x|^{\alpha-\eta}\phi(x) - |y|^{\alpha-\eta}\phi(y)\,|}{|x-y|^\alpha} < \infty \right\}.$$

Equipped with the norm $||\phi||_{\alpha,\eta,\,I}$, the space $\mathcal{C}^\alpha(I)_\eta$ is a Banach space. When I is an interval such as $I = [b,\infty)$, we omit the bracket of $\mathcal{C}^\alpha(I)_\eta$ such as $\mathcal{C}^\alpha[b,\infty)_\eta$.

The space $\mathcal{C}^\alpha(I)_\eta$ is a generalization of the standard Hölder space, namely, a function space of continuous functions such that $|\phi(x) - \phi(y)| \lesssim |x-y|^\alpha$. In the case where $I = [a,b]$ with $0 < a < b < \infty$, the space $\mathcal{C}^\alpha[a,b]_\alpha$ is no other than the standard Hölder space, and $\mathcal{C}^\alpha[a,b]_\eta$, $\eta \in \mathbb{R}$, is isomorphic to the standard Hölder space.

In analogy to the standard Hölder space, $\mathcal{C}^\alpha(I)_\eta$ has a canonical inclusion:

Lemma 2.1. *Let $\eta \in \mathbb{R}$ and let $I \subset \mathbb{R}$.*

(1) *If $0 < \alpha \le \beta \le 1$ then there is a bounded inclusion $\mathcal{C}^\beta(I)_\eta \hookrightarrow \mathcal{C}^\alpha(I)_\eta$. In particular,*

$$\lim_{R\to\infty} ||\phi||_{\beta,\eta,\,(R,\infty)} = 0 \implies \lim_{R\to\infty} ||\phi||_{\alpha,\eta,\,(R,\infty)} = 0.$$

(2) *Provided I is bounded, if $0 < \alpha \le \beta \le 1$, $\eta \le \vartheta$ then $\mathcal{C}^\beta(I)_\vartheta \hookrightarrow \mathcal{C}^\alpha(I)_\eta$.*

(3) *If ϕ is a continuously differentiable function with the asymptotic behavior*

$$\phi(x) = Lx^\eta[1+o(1)], \quad \phi'(x) = L\eta x^{\eta-1}[1+o(1)], \quad x \to \infty,$$

then, for each $\alpha \in (0,1]$,

$$\lim_{R\to\infty} ||\phi(x) - Lx^\eta||_{\alpha,\eta,\,(R,\infty)} = 0.$$

Proof. We shall prove the lemma only in the case where $I \subset (0,\infty)$, since other cases can be treated in a similar manner. For the proof of (1), it suffices to prove that $\mathcal{C}^\beta(I)_0 \subset \mathcal{C}^\alpha(I)_0$ because $\phi \in \mathcal{C}^\alpha(I)_\eta$ if and only if $x^{-\eta}\phi(x) \in \mathcal{C}^\alpha(I)_0$. Let $\phi \in \mathcal{C}^\beta(I)_0$, $0 < \alpha \le \beta \le 1$. Then, for $x, y \in I$, $y \le x$, we get

$$\begin{aligned}
&|x^\alpha\phi(x) - y^\alpha\phi(y)| \\
&= |x^{\alpha-\beta}(x^\beta\phi(x) - y^\beta\phi(y)) + x^{\alpha-\beta}(y^\beta - x^\beta)\phi(y) + (x^\alpha - y^\alpha)\phi(y)| \\
&\le 2x^{\alpha-\beta}|x-y|^\beta\, ||\phi||_{\beta,0,\,I} + |x-y|^\alpha\, ||\phi||_{\beta,0,\,I} \\
&\le 2\,|1-(y/x)|^{\beta-\alpha}|x-y|^\alpha\, ||\phi||_{\beta,0,\,I} + |x-y|^\alpha\, ||\phi||_{\beta,0,\,I} \\
&\le 3\,|x-y|^\alpha\, ||\phi||_{\beta,0,\,I}.
\end{aligned}$$

This means that $||\phi||_{\alpha,0,\,I} \leq 3||\phi||_{\beta,0,\,I}$. Hence we find $\phi \in \mathcal{C}^{\alpha}(I)_0$, and show assertion (1). Assertion (2) can be proved in a similar manner to the above.

By (1), for the proof of (3), it is enough to show that $||\phi(x) - Lx^{\eta}||_{1,\eta,\,(R,\infty)} \to 0$ as $R \to \infty$. Noting that $||\phi(x) - Lx^{\eta}||_{1,\eta,\,(R,\infty)} = ||x^{-\eta}\phi(x) - L||_{1,0,\,(R,\infty)}$, we compute

$$\begin{aligned}
&x(x^{-\eta}\phi(x) - L) - y(y^{-\eta}\phi(y) - L)\\
&= \int_y^x (\xi(\xi^{-\eta}\phi(\xi) - L))' d\xi\\
&= \int_y^x (\xi^{-\eta}\phi(\xi) - L) d\xi + \int_y^x \xi(\xi^{-\eta}\phi(\xi))' d\xi.
\end{aligned}$$

But, by assumption,

$$\begin{aligned}
&\xi^{-\eta}\phi(\xi) - L = o(1),\\
&\xi(\xi^{-\eta}\phi(\xi))' = -\eta\xi^{-\eta}\phi(\xi) + \xi^{1-\eta}\phi'(\xi) = -\eta(L + o(1)) + L\eta + o(1) = o(1),
\end{aligned}$$

as $\xi \to \infty$, and hence, $|x(x^{-\eta}\phi(x) - L) - y(y^{-\eta}\phi(y) - L)| \leq M|x - y|\, o(1)$, as $R \to \infty$, with some constant M. This proves assertion (3). $\square$

A scaling operator is useful in our analysis for integral equation (1.2): let $\eta \in \mathbb{R}$, $R > 0$ and let S_R^{η} be an operator defined by

$$S_R^{\eta}\phi(x) = R^{-\eta}\phi(Rx). \tag{2.1}$$

The operator S_R^{η} transposes a function ϕ defined on the interval (R, ∞) to a function $R^{-\eta}\phi(Rx)$ on the interval $(1, \infty)$. Clearly, for a function $\phi_0(x) := Lx^{\eta}$ with $L \in \mathbb{R}$,

$$S_R^{\eta}\phi_0 = \phi_0, \tag{2.2}$$

where ϕ_0 in the left-hand side is regarded as the function Lx^{η} on (R, ∞) and ϕ_0 in the right-hand side is regarded as the the function Lx^{η} on $(1, \infty)$. It is also clear from the definition (2.1) that

$$S_{R_1}^{\eta} S_{R_2}^{\eta} = S_{R_1 R_2}^{\eta}. \tag{2.3}$$

A characteristic of the scaling operator S_R^{η} is the isometry in the following sense:

Lemma 2.2. *For each $\alpha \in (0, 1]$, S_R^{η} is an isometric operator from $\mathcal{C}^{\alpha}(R, \infty)_{\eta}$ onto $\mathcal{C}^{\alpha}(1, \infty)_{\eta}$. That is, $||S_R^{\eta}\phi||_{\alpha,\eta,\,(1,\infty)} = ||\phi||_{\alpha,\eta,\,(R,\infty)}$.*

Proof. For $\phi \in \mathcal{C}^\alpha(R,\infty)_\eta$,

$$\begin{aligned}
&||S_R^\eta \phi||_{\alpha,\eta,\,(1,\infty)} \\
&\quad = \sup_{1<x<\infty} |x^{-\eta}(S_R^\eta\phi)(x)| + \sup_{\substack{1<x,y<\infty \\ x\neq y}} \frac{|x^{\alpha-\eta}(S_R^\eta\phi)(x) - y^{\alpha-\eta}(S_R^\eta\phi)(y)|}{|x-y|^\alpha} \\
&\quad = \sup_{1<x<\infty} |(Rx)^{-\eta}\phi(Rx)| + \sup_{\substack{1<x,y<\infty \\ x\neq y}} \frac{|x^{\alpha}(Rx)^{-\eta}\phi(Rx) - y^{\alpha}(Ry)^{-\eta}\phi(Ry)|}{|x-y|^\alpha} \\
&\quad = \sup_{R<x<\infty} |x^{-\eta}\phi(x)| + \sup_{\substack{R<x,y<\infty \\ x\neq y}} \frac{|x^{\alpha-\eta}\phi(x) - y^{\alpha-\eta}\phi(y)|}{|x-y|^\alpha} = ||\phi||_{\alpha,\eta,\,(R,\infty)}.
\end{aligned}$$

This proves the lemma. □

Lemma 2.3. *Let $\sigma > 0$, $0 < \alpha < \frac{1}{2}$, $R > 0$. Then we have the following, commutative diagram:*

$$\begin{array}{ccc}
\mathcal{C}^\alpha(R,\infty)_{1+\sigma} & \xrightarrow{\quad\mathcal{B}\quad} & \mathcal{C}^{\alpha+\frac{1}{2}}(R,\infty)_{-\frac{\sigma}{2}} \\
S_R^{1+\sigma}\Big\downarrow\cong & & S_R^{-\frac{\sigma}{2}}\Big\downarrow\cong \\
\mathcal{C}^\alpha(1,\infty)_{1+\sigma} & \xrightarrow{\quad\mathcal{B}\quad} & \mathcal{C}^{\alpha+\frac{1}{2}}(1,\infty)_{-\frac{\sigma}{2}}
\end{array}$$

Here the vertical arrows in the diagram are homeomorphisms.

Proof. We first note that the left-hand side of (1.2) gives an explicit expression of the blow-up time map $\mathcal{B}$:

$$\mathcal{B}f(h) = \frac{1}{\sqrt{2}}\int_h^\infty \frac{du}{\sqrt{\int_h^u f(\xi)d\xi}}. \tag{2.4}$$

As was shown in [13, Propositions 2.2 and 3.1], $\mathcal{B}f$ belongs to $\mathcal{C}^{\alpha+\frac{1}{2}}(R,\infty)_{-\frac{\sigma}{2}}$ if f belongs to $\mathcal{C}^\alpha(R,\infty)_{1+\sigma}$. Noting that (2.4) can be rewritten as

$$\mathcal{B}f(h) = \sqrt{\frac{h}{2}}\int_1^\infty \frac{dr}{\sqrt{\int_1^r f(hs)ds}},$$

we have, for $1 < h < \infty$,

$$
\begin{aligned}
S_R^{-\frac{\sigma}{2}} \mathcal{B} f(h) &= R^{\frac{\sigma}{2}} (\mathcal{B} f)(Rh) = R^{\frac{\sigma}{2}} \sqrt{\frac{Rh}{2}} \int_1^\infty \frac{dr}{\sqrt{\int_1^r f(Rhs)ds}} \\
&= \sqrt{\frac{h}{2}} \int_1^\infty \frac{dr}{\sqrt{\int_1^r R^{-1-\sigma} f(Rhs)ds}} = \sqrt{\frac{h}{2}} \int_1^\infty \frac{dr}{\sqrt{\int_1^r S_R^{1+\sigma} f(hs)ds}} \\
&= \mathcal{B} S_R^{1+\sigma} f(h).
\end{aligned}
$$

This yields the commutative diagram in the lemma. In view of Lemma 2.2, vertical arrows in the diagram are homeomorphisms. □

§ 2.2. Local existence

One of typical super-linear nonlinear terms is $f_0(u) = cu^{1+\sigma}$, $c, \sigma > 0$, which is defined for $u > 0$. For this function $f_0(u)$, the blow-up time function is calculated as

$$T_0(h) = c' h^{-\frac{\sigma}{2}},$$

where

$$c' = \frac{1}{\sqrt{2c(2+\sigma)}} B\left(\frac{\sigma}{2(2+\sigma)}, \frac{1}{2}\right), \tag{2.5}$$

with the beta function $B(\cdot,\cdot)$. Around this correspondence

$$\mathcal{B} : f_0(u) = cu^{1+\sigma} \mapsto T_0(h) = c' h^{-\frac{\sigma}{2}}, \tag{2.6}$$

the inverse time map $\mathcal{B}$ gives a local homeomorphism in the following sense:

Theorem 2.4 ([13]). *Let α be any number fixed such that $0 < \alpha < \frac{1}{2}$. Then $\mathcal{B}$ maps a sufficiently small neighborhood of f_0 in $\mathcal{C}^\alpha(1,\infty)_{1+\sigma}$ homeomorphically onto a neighborhood of T_0 in $\mathcal{C}^{\alpha+\frac{1}{2}}(1,\infty)_{-\frac{\sigma}{2}}$.*

From this local theorem in a framework of function spaces, we can draw a local existence theorem of a more concrete form:

Theorem 2.5 (Local existence). *Let T be a prescribed function satisfying*

$$\lim_{R\to\infty} ||T - T_0||_{\alpha+\frac{1}{2},\, -\frac{\sigma}{2},\, (R,\infty)} = 0, \tag{2.7}$$

where $\sigma > 0$, $\alpha \in (0, \frac{1}{2})$. Then $\mathcal{B} f = T$ has a positive solution $f \in \mathcal{C}^\alpha(H,\infty)_{1+\sigma}$ on some interval (H,∞) (with sufficiently large H) satisfying

$$\lim_{R\to\infty} ||f - f_0||_{\alpha,\, 1+\sigma,\, (R,\infty)} = 0. \tag{2.8}$$

Proof. By Theorem 2.4, there exist an open ball

$$U(f_0) = \{f : ||f - f_0||_{\alpha.\,1+\sigma,\,(1,\infty)} < \delta\} \tag{2.9}$$

in $\mathcal{C}^{\alpha}(1,\infty)_{1+\sigma}$ and an open neighborhood $V(T_0)$ of T_0 in $\mathcal{C}^{\alpha+\frac{1}{2}}(1,\infty)_{-\frac{\sigma}{2}}$ such that $\mathcal{B}$ maps homeomorphically $U(f_0)$ onto $V(T_0)$.

We denote a restriction of a prescribed function T on (H,∞) by (the same letter) T for brevity. Then, for each $H \geq 1$,

$$S_H^{-\frac{\sigma}{2}} T - T_0 = S_H^{-\frac{\sigma}{2}}(T - T_0),$$

because $S_H^{-\frac{\sigma}{2}} T_0 = T_0$ by (2.2). Hence, by Lemma 2.2,

$$||S_H^{-\frac{\sigma}{2}} T - T_0||_{\alpha+\frac{1}{2},\,-\frac{\sigma}{2},\,(1,\infty)} = ||T - T_0||_{\alpha+\frac{1}{2},\,-\frac{\sigma}{2},\,(H,\infty)}. \tag{2.10}$$

This, combined with the assumption (2.7), shows that, if H is sufficiently large then $S_H^{-\frac{\sigma}{2}} T$ belongs to the neighborhood $V(T_0)$. Therefore there exists a unique $f_H \in U(f_0)$ such that $\mathcal{B} f_H = S_H^{-\frac{\sigma}{2}} T$. Notice that $||f_H - f_0||_{\alpha.\,1+\sigma,\,(1,\infty)} < \delta$.

$$\begin{tikzcd}
f \in \mathcal{C}^{\alpha}(H,\infty)_{1+\sigma} \arrow[r, "\mathcal{B}"] \arrow[d, "S_H^{1+\sigma}"', "\cong"] & \mathcal{C}^{\alpha+\frac{1}{2}}(H,\infty)_{-\frac{\sigma}{2}} \ni T \arrow[d, "S_H^{-\frac{\sigma}{2}}"', "\cong"] \\
f_H \in U(f_0) \arrow[r, "\mathcal{B}"] & V(T_0) \ni S_H^{-\frac{\sigma}{2}} T.
\end{tikzcd}$$

We now define a function $f \in \mathcal{C}^{\alpha}(H,\infty)_{1+\sigma}$ by

$$f = (S_H^{1+\sigma})^{-1} f_H \tag{2.11}$$

(see the above diagram). Then, by Lemma 2.3,

$$\mathcal{B} f = (S_H^{-\frac{\sigma}{2}})^{-1} \mathcal{B} S_H^{1+\sigma} f = (S_H^{-\frac{\sigma}{2}})^{-1} \mathcal{B} f_H = T.$$

This proves that f is a solution of $\mathcal{B} f = T$ on (H,∞).

For $R > H$, in a similar manner to for f_H, we can define $f_R \in U(f_0)$ so that

$$(S_R^{1+\sigma})^{-1} f_R \in \mathcal{C}^{\alpha}(R,\infty)_{1+\sigma}, \quad \mathcal{B} f_R = S_R^{-\frac{\sigma}{2}} T.$$

In order to show that f defined by (2.11) satisfies (2.8), we shall prove that

$$(S_R^{1+\sigma})^{-1} f_R = f \ \text{ for } \ R > H, \tag{2.12}$$

in other words, $(S_R^{1+\sigma})^{-1}f_R$ is just a restriction on (R,∞) of f. To this end, let $\underline{f}$ be a restriction on (R,∞) of the function $f \in \mathcal{C}^\alpha(H,\infty)_{1+\sigma}$. Then $S_H^{1+\sigma}\underline{f}$ is a restriction on $(R/H,\infty)$ of $f_H \in U(f_0)$, and so, $\mathcal{B}S_H^{1+\sigma}\underline{f} = \mathcal{B}f_H = S_H^{-\frac{\sigma}{2}}T$ in the interval $(R/H,\infty)$. This, together with (2.3), shows that

$$\mathcal{B}S_H^{1+\sigma}\underline{f} = \mathcal{B}f_H = S_H^{-\frac{\sigma}{2}}T = S_{H/R}^{-\frac{\sigma}{2}}S_R^{-\frac{\sigma}{2}}T$$

in the interval $(R/H,\infty)$. Here T is viewed as a restriction on (R,∞) of the prescribed function T.

Accordingly, by Lemma 2.3,

$$\mathcal{B}(S_{H/R}^{1+\sigma})^{-1}S_H^{1+\sigma}\underline{f} = (S_{H/R}^{-\frac{\sigma}{2}})^{-1}\mathcal{B}S_H^{1+\sigma}\underline{f} = S_R^{-\frac{\sigma}{2}}T.$$

The function $(S_{H/R}^{1+\sigma})^{-1}S_H^{1+\sigma}\underline{f}$ belongs to $U(f_0)$, because, by (2.2) and Lemma 2.2,

$$\begin{aligned}&||(S_{H/R}^{1+\sigma})^{-1}S_H^{1+\sigma}\underline{f} - f_0||_{\alpha.\,1+\sigma,\,(1,\infty)} = ||(S_{H/R}^{1+\sigma})^{-1}(S_H^{1+\sigma}\underline{f} - f_0)||_{\alpha.\,1+\sigma,\,(1,\infty)}\\&= ||S_H^{1+\sigma}\underline{f} - f_0||_{\alpha.\,1+\sigma,\,(R/H,\infty)} = ||f_H - f_0||_{\alpha.\,1+\sigma,\,(R/H,\infty)}\\&\le ||f_H - f_0||_{\alpha.\,1+\sigma,\,(1,\infty)} < \delta.\end{aligned}$$

Consequently, we have

$$\mathcal{B}(S_{H/R}^{1+\sigma})^{-1}S_H^{1+\sigma}\underline{f} = S_R^{-\frac{\sigma}{2}}T, \quad (S_{H/R}^{1+\sigma})^{-1}S_H^{1+\sigma}\underline{f} \in U(f_0).$$

Comparing this with

$$\mathcal{B}f_R = S_R^{-\frac{\sigma}{2}}T, \quad f_R \in U(f_0),$$

and noting $\mathcal{B} : U(f_0) \to V(T_0)$ is injective, we conclude that

$$f_R = (S_{H/R}^{1+\sigma})^{-1}S_H^{1+\sigma}\underline{f}.$$

Therefore, by (2.3),

$$(S_R^{1+\sigma})^{-1}f_R = (S_R^{1+\sigma})^{-1}(S_{H/R}^{1+\sigma})^{-1}S_H^{1+\sigma}\underline{f} = (S_H^{1+\sigma})^{-1}S_H^{1+\sigma}\underline{f} = \underline{f}.$$

Thus we have proved (2.12).

It follows from (2.7), (2.10) that $S_R^{-\frac{\sigma}{2}}T \to T_0$ in the norm of $\mathcal{C}^{\alpha+\frac{1}{2}}(1,\infty)_{-\frac{\sigma}{2}}$ as $R \to \infty$. Since $\mathcal{B}f_R = S_R^{-\frac{\sigma}{2}}T$, $\mathcal{B}f_0 = T_0$ and the inverse of $\mathcal{B} : U(f_0) \to V(T_0)$ is continuous, this implies that

$$\lim_{R\to\infty} ||f_R - f_0||_{\alpha,\,1+\sigma,\,(1,\infty)} = 0.$$

But, by (2.11), (2.12), (2.2) and Lemma 2.2,

$$\begin{aligned}&||f - f_0||_{\alpha,\,1+\sigma,\,(R,\infty)} = ||(S_R^{1+\sigma})^{-1}f_R - f_0||_{\alpha,\,1+\sigma,\,(R,\infty)}\\&= ||(S_R^{1+\sigma})^{-1}(f_R - f_0)||_{\alpha,\,1+\sigma,\,(R,\infty)} = ||f_R - f_0||_{\alpha,\,1+\sigma,\,(1,\infty)}.\end{aligned}$$

Accordingly the solution f defined by (2.11) satisfies (2.8). Thus we have proved that there exists $H \geq 1$ such that $\mathcal{B}f = T$ has a solution f on (H, ∞) satisfying (2.12). $\square$

We draw the following conclusion from Theorem 2.5:

Corollary 2.6. *Given a positive $T \in C^1(a_0, \infty)$ having the asymptotic behavior*

$$T(h) = c'\, h^{-\frac{\sigma}{2}}[1 + o(1)], \quad T'(h) = -\frac{\sigma}{2}\, c'\, h^{-\frac{\sigma}{2}-1}[1 + o(1)], \quad h \to \infty, \tag{2.13}$$

where $c' > 0$, the equation $\mathcal{B}f = T$ has a positive, continuous solution f on some interval (H, ∞) having the asymptotic behavior

$$f(u) = c\, u^{1+\sigma}\, [1 + o(1)], \quad u \to \infty, \tag{2.14}$$

where c is determined from c' by (2.5).

Proof. To prove the corollary, it is enough to show that a function T with the behavior (2.13) satisfies the assumption (2.7). But this is the case for which Lemma 2.1(3) with $\eta = -\frac{\sigma}{2}$, $\phi = T$, $Lx^\eta = c'h^{-\frac{1}{2}}$, and $\alpha + \frac{1}{2}$ instead of α is applicable. $\square$

§ 2.3. Local uniqueness

Based upon Theorem 2.4, we get the following uniqueness.

Theorem 2.7 (Local uniqueness). *Let $0 < \alpha < \frac{1}{2}$ and let $f_0(u) = cu^{1+\sigma}$ with $c, \sigma > 0$. Then, given a positive function T on (a_0, ∞) satisfying* (2.7)*, a positive solution f of $\mathcal{B}f = T$ which satisfies* (2.8) *is unique.*

Proof. As in the proof of Theorem 2.5, let $U(f_0)$ be an open ball in $\mathcal{C}^\alpha(1, \infty)_{1+\sigma}$ defined in (2.9), which is mapped by $\mathcal{B}$ homeomorphically to an open neighborhood $V(T_0)$ in $\mathcal{C}^{\alpha+\frac{1}{2}}(1, \infty)_{-\frac{\sigma}{2}}$ of $T_0 = c'h^{-\frac{\sigma}{2}}$ with c' determined by (2.5).

We suppose that, as well as f, a function g is a solution of $\mathcal{B}f = T$ satisfying the condition (2.8), that is,

$$\lim_{R\to\infty} ||g - f_0||_{\alpha,\, 1+\sigma,\, (R,\infty)} = 0.$$

Since, by (2.2) and lemma 2.2,

$$||S_R^{1+\sigma} f - f_0||_{\alpha,\, 1+\sigma,\, (1,\infty)} = ||S_R^{1+\sigma}(f - f_0)||_{\alpha,\, 1+\sigma,\, (1,\infty)} = ||f - f_0||_{\alpha,\, 1+\sigma,\, (R,\infty)},$$

it follows from (2.12) that $||f - f_0||_{\alpha,\, 1+\sigma,\, (R,\infty)} < \delta$ for sufficiently large R. Hence $S_R^{1+\sigma} f \in U(f_0)$ for sufficiently large R. Similarly $S_R^{1+\sigma} g \in U(f_0)$ for sufficiently large R. Since $\mathcal{B}f = \mathcal{B}g$ on the interval (R, ∞), by Lemma 2.3, we have

$$\mathcal{B}S_R^{1+\sigma} f = S_R^{-\frac{\sigma}{2}} \mathcal{B}f = S_R^{-\frac{\sigma}{2}} \mathcal{B}g = \mathcal{B}S_R^{1+\sigma} g.$$

This implies that $S_R^{1+\sigma} f = S_R^{1+\sigma} g$ because of the injectivity of $\mathcal{B} : U(f_0) \to V(T_0)$. Hence $f = g$ as an element in $\mathcal{C}^\alpha(R, \infty)_{1+\sigma}$, and so, $f(u) = g(u)$ for any $u \in (R, \infty)$. The proof is complete. $\square$

Theorem 2.7 reflects well a local one-to-one correspondence of the map $\mathcal{B}$, originally given by Theorem 2.4. However, it is somewhat weak for answering a fundamental question such as whether the base function $f_0(u) = cu^{1+\sigma}$ is the only solution of $\mathcal{B}f = T_0$. When we impose a stronger assumption (2.13) on T, we can answer this question in a more natural way:

Theorem 2.8 ([14])**.** *Let T be a C^1-function with the behavior* (2.13)*. Then the solution f, which exists by Corollary* 2.6*, is the only solution of $\mathcal{B}f = T$ that has the asymptotic behavior* (2.14)*.*

So far we have confined ourselves to the correspondence (2.6). We here announce two results around a correspondence

$$\mathcal{B} : f_0(u) = c\, e^u \;\mapsto\; T_0(h) = c' e^{-\frac{h}{2}}, \tag{2.15}$$

with $c' = \frac{\pi}{\sqrt{2c}}$, which is associated with another typical super-linear nonlinear term $f_0(u) = c\, e^u$.

The first one is along a similar analysis to that deducing Theorems 2.4, 2.5, 2.7:

Theorem 2.9. *Let T be a prescribed function satisfying*

$$\lim_{R\to\infty} ||T(-\log x) - T_0(-\log x)||_{\alpha+\frac{1}{2},\, \frac{1}{2},\, (0, \frac{1}{R})} = 0,$$

where $\alpha \in (0, \frac{1}{2})$. Then $\mathcal{B}f = T$ has a unique positive solution on some interval (H, ∞) (with sufficiently large H) that satisfies

$$\lim_{R\to\infty} ||f(-\log x) - f_0(-\log x)||_{\alpha,\, -1,\, (0, \frac{1}{R})} = 0.$$

The second one is along a similar strategy to that leading us to Corollary 2.6 and Theorem 2.8:

Theorem 2.10. *Given a positive $T \in C^1(a_0, \infty)$ having the asymptotic behavior*

$$T(h) = c'\, e^{-\frac{h}{2}}[1 + o(1)], \quad T'(h) = -\frac{c'}{2}\, e^{-\frac{h}{2}}[1 + o(1)], \quad h \to \infty,$$

where $c' > 0$, the equation $\mathcal{B}f = T$ has a unique positive, continuous solution f on some interval (H, ∞) that has the asymptotic behavior

$$f(u) = c\, e^u\, [1 + o(1)], \quad u \to \infty,$$

where c is a positive number determined from c' by $c' = \frac{\pi}{\sqrt{2c}}$.

§ 3. Global theory

In general, it is exceedingly difficult to expand a global theory assuring global existence and uniqueness of a solution obtained near an initial point. In this section, we show that our inverse problem has nice features: under a simple assumption on T, continuation of the solution obtained near our initial point, that is $+\infty$, to the left can be insured globally as well as its global uniqueness. Throughout this section we assume that a solution of $\mathcal{B}f = T$ exists on $[H, \infty)$, where H is supposed to be large. Notice that the difference between $[H, \infty)$ and (H, ∞), the latter is used in the previous section, causes no concern, since we can replace H to be a bit larger than previous H.

§ 3.1. Global uniqueness

A global uniqueness of solutions is automatically guaranteed by the following:

Theorem 3.1 ([14]). *Let f, g be positive, continuous solutions of $\mathcal{B}f = T$ on the interval (a_0, ∞). If $f(u) = g(u)$ on some interval $[H, \infty)$ then $f(u) = g(u)$ on (a_0, ∞).*

This result, which means that an initial solution near $+\infty$ can not bifurcate at any point in (a_0, ∞), is vital; each local uniqueness result obtained (for example, that in §2.3) can be recast as a global uniqueness result by this theorem. For instance, Theorem 2.8 can be recast as:

Theorem 3.2. *Let $T \in C^1(a_0, \infty)$ be a positive function having the behavior*

$$T(h) = c'\, h^{-\frac{\sigma}{2}}[1 + o(1)], \quad T'(h) = -\frac{\sigma}{2}\, c'\, h^{-\frac{\sigma}{2}-1}[1 + o(1)], \quad h \to \infty.$$

Then a positive, continuous solution f of $\mathcal{B}f = T$ having the asymptotic behavior

$$f(u) = c\, u^{1+\sigma}\, [1 + o(1)], \quad u \to \infty,$$

is the only positive, continuous solution on (a_0, ∞) of $\mathcal{B}f = T$ which has this behavior.

Remark. The positive constant c in the above theorem is uniquely determined from the constant c' in the theorem, since, if a solution $f \in C[H, \infty)$ of $\mathcal{B}f = T$ has the behavior $f(u) = cu^{1+\sigma}[1 + o(1)]$, $u \to \infty$, then, it is readily seen from (1.2) and the Lebesgue's dominated convergence theorem that

$$T(h) = \frac{1}{\sqrt{2}} \int_h^\infty \frac{du}{\sqrt{\int_h^u c\,\xi^{1+\sigma}[1 + o(1)]d\xi}} = c'\, h^{-\frac{\sigma}{2}}[1 + o(1)], \quad h \to \infty.$$

Consequently, c is uniquely determined, which must be the constant determined from c' via (2.5).

We here present an example that illustrates an application of Theorem 3.2:

Example 3.3. In what follows, $K(k)$ denotes the complete elliptic function of the first kind. Let $f(u) = u^2 + 1$. Then $T = \mathcal{B}f$ is computed by (1.2) as

$$T(h) = \frac{1}{\sqrt{2}} \int_h^\infty \frac{du}{\sqrt{\int_h^u (\xi^2+1)d\xi}} = \sqrt{\frac{3}{2}} \int_h^\infty \frac{du}{\sqrt{(u-h)(u^2+hu+h^2+3)}}$$
$$= \sqrt{\frac{6}{\pm h}} \int_0^\infty \frac{ds}{\sqrt{s^4+3s^2+3\left(1+\frac{1}{h^2}\right)}} = \frac{\sqrt{2}\, 3^{\frac{1}{4}}}{(h^2+1)^{\frac{1}{4}}} K\left(\sqrt{\frac{1}{2} - \frac{\sqrt{3}}{4}\frac{h}{\sqrt{h^2+1}}}\right).$$

Here we have used the substitution $u = h(1 \pm s^2)$, corresponding to $\pm h > 0$ and the formula

$$\int_0^\infty \frac{ds}{\sqrt{s^4 + As^2 + B^2}} = \frac{1}{\sqrt{B}} K\left(\sqrt{\frac{1}{2} - \frac{A}{4B}}\right), \quad B > 0,\, 2B > A.$$

Thus, with $a_0 = -\infty$, we have

$$f(u) := u^2 + 1 \overset{\mathcal{B}}{\longmapsto} T(h) := \frac{\sqrt{2}\, 3^{\frac{1}{4}}}{(h^2+1)^{\frac{1}{4}}} K\left(\sqrt{\frac{1}{2} - \frac{\sqrt{3}}{4}\frac{h}{\sqrt{h^2+1}}}\right), \quad -\infty < u, h < \infty.$$

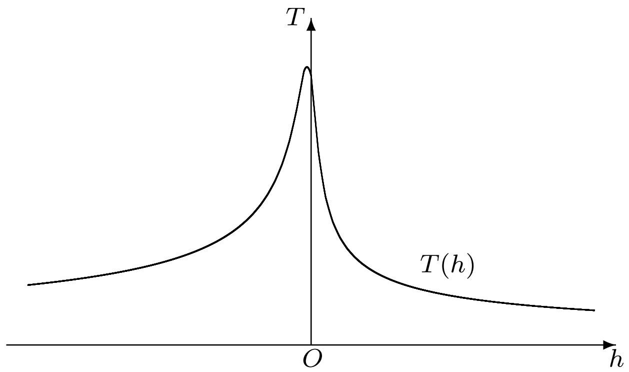

Figure 3. The function T.

As in a situation stated just before Theorem 2.8, whether a pull-back $\mathcal{B}^{-1}(T)$ is occupied by only $u^2 + 1$, in other words, whether $u^2 + 1$ is the only nonlinear term realizing this blow-up time function T, is not trivial at all. Since

$$T(h) = c'\, h^{-\frac{1}{2}}[1 + o(1)], \quad T'(h) = -\frac{1}{2}\, c'\, h^{-\frac{3}{2}}[1 + o(1)], \quad h \to \infty,$$

where $c' = \frac{1}{\sqrt{6}} B(\frac{1}{6}, \frac{1}{2})$ (see Figure 3 for the graph of T), we can apply Theorem 3.2 with $\sigma = 1$ to conclude that the pull-back is occupied by only f, if we restrict the set on a set of functions having the behavior $f(u) = u^2[1 + o(1)]$ as $u \to \infty$.

§ 3.2. Global existence

We turn to the question of how large the domain of solution may be. This question asks under what condition on T a solution f near the infinity can be continued to the left. Before answering this question, we point out that the blow-up time map $\mathcal{B}$ improves the Hölder continuity of functions by the index $\frac{1}{2}$ in the following sense.

Proposition 3.4. *If $f = f(u)$ is a positive, locally Hölder continuous function on (a_0, ∞) with the Hölder exponent $\alpha \in [0, \frac{1}{2})$ for which the integral in* (1.2) *is finite for each $h > a_0$, then $T = \mathcal{B}f$ is a locally Hölder continuous function on (a_0, ∞) with the Hölder exponent $\alpha + \frac{1}{2}$.*

A particular case $\alpha = 0$ in the above proposition shows that, if a positive, continuous solution f of $\mathcal{B}f = T$ can be continued up to a point $a > a_0$ preserving its positivity and continuity, then $\mathcal{B}f$ is necessarily Hölder continuous with the Hölder exponent $\frac{1}{2}$ even at a. In other words, if a prescribed function T is not Hölder continuous with the Hölder exponent $\frac{1}{2}$ at a, then $\mathcal{B}f = T$ does not admit a solution f that is still positive and continuous at a. Thus, for establishing a general, continuation result, we must assume the Hölder continuity of T with the Hölder exponent $\frac{1}{2}$ at least.

The following result implies that Hölder continuity of T with the Hölder exponent 1 is sufficient to insure the continuation to the left of a solution.

Theorem 3.5 ([14]). *Assume that T is a prescribed, positive, continuous function on (a_0, ∞). If f is a positive, continuous solution of $\mathcal{B}f = T$ on an interval $[H, \infty)$ with $H > a_0$ then the solution f can be uniquely continued to the left as long as T is locally Lipschitz continuous. In addition, the solution is necessarily* (*locally*) *Hölder continuous with the Hölder exponent α, an arbitrary number less than $\frac{1}{2}$.*

Remark. The uniqueness of this continuation is guaranteed by Theorem 3.1.

Example 3.6. Let

$$T(h) = \frac{1}{1 + |h|^{\frac{\sigma}{2}}}, \quad -\infty < h < \infty,$$

where $\sigma > 0$. This function is of the class C^1 on $(0, \infty)$ with the asymptotic behavior (2.13) where $c' = 1$. Therefore, in view of Corollary 2.6, the equation $\mathcal{B}f = T$ admits a positive, continuous solution $f(u)$ near $+\infty$ with the asymptotic behavior $f(u) = cu^{1+\sigma}[1+o(1)]$ as $u \to \infty$. By Theorem 3.2, this is the only positive, continuous solution having this asymptotic behavior. Since the function T is locally Lipschitz continuous on $(0, \infty)$, this solution found near $+\infty$ can be continued in $(0, \infty)$ by virtue of Theorem 3.5.

If $\sigma \geq 2$ then the function T is locally Lipschitz continuous on the whole interval $(-\infty, \infty)$, because

$$T(h) - T(0) = -\frac{|h|^{\frac{\sigma}{2}}}{1 + |h|^{\frac{\sigma}{2}}}.$$

Hence, by Theorem 3.5, the solution found near $+\infty$ can be continued beyond 0 to the left without end, in the case $\sigma \geq 2$. On the other hand, the solution can not be continued up to 0 in the case $\sigma < 1$ for the reason stated immediately after Proposition 3.4.

Assertion at the end of Theorem 3.5 presents a kind of converse to Proposition 3.4 in the sense that the inverse $\mathcal{B}^{-1}$ of the blow-up time map gives a correspondence from a set of locally Lipschitz continuous functions, say C^{1-0}, to a set of locally Hölder continuous functions with any Hölder exponent less than $\frac{1}{2}$, say $C^{\frac{1}{2}-0}$, that is, $\mathcal{B}^{-1}(C^{1-0}) \subset C^{\frac{1}{2}-0}$. From this observation, it might be expected that a preciser converse $\mathcal{B}^{-1}(C^{\beta}) \subset C^{\beta - \frac{1}{2}}$ holds for each $\beta \in (\frac{1}{2}, 1)$, or, that the assumption that T is locally Hölder continuous with the Hölder exponent 1 in Theorem 3.5, can be relaxed to a weaker smoothness with the Hölder exponent $\beta < 1$. But, against expectation, the conclusion of Theorem 3.5 is no longer valid if we replace the Hölder exponent 1 by $\beta < \frac{2}{3}$, as is shown in the following example.

Example 3.7. Let $0 < \sigma < 2$ and set

$$f(u) = Au^{\frac{\sigma}{2}-1} + Bu^{1+\sigma}, \quad u > 0,$$

where A, B are positive constants. Because of $\sigma < 2$, $f(u)$ is going to $+\infty$ as $u \to 0$. We will explore a behavior of a blow-up time function $T(h)$ as $h \to 0$. Since a choice of A, B has nothing to do with portrait (see Figure 4) of the behavior, we fix A, B as $A = \frac{\sigma}{4}$, $B = \frac{2+\sigma}{2}$. Then

$$T(h) := \mathcal{B}f(h) = \int_h^\infty \frac{du}{\sqrt{u^{\frac{\sigma}{2}} - h^{\frac{\sigma}{2}} + u^{2+\sigma} - h^{2+\sigma}}}.$$

Clearly $T(h)$ is a positive, continuous function on $[0, \infty)$. To find the Hölder exponent of $T(h)$ at $h = 0$, we rewrite $T(h)$ as

$$T(h) = \int_0^\infty \frac{du}{\sqrt{u^{\frac{\sigma}{2}} + u^{2+\sigma}}} - \int_0^h \frac{du}{\sqrt{u^{\frac{\sigma}{2}} + u^{2+\sigma}}} + \int_h^\infty \left(\frac{1}{\sqrt{u^{\frac{\sigma}{2}} - h^{\frac{\sigma}{2}} + u^{2+\sigma} - h^{2+\sigma}}} - \frac{1}{\sqrt{u^{\frac{\sigma}{2}} + u^{2+\sigma}}} \right) du. \tag{3.1}$$

The first and the second terms in the right-hand side can be computed as

$$\int_0^\infty \frac{du}{\sqrt{u^{\frac{\sigma}{2}} + u^{2+\sigma}}} = \frac{2}{4+\sigma} B\left(\frac{\sigma}{4+\sigma}, \frac{4-\sigma}{8+2\sigma}\right),$$

via the substitution $\frac{1}{s} = 1 + u^{2+\frac{\sigma}{2}}$, and

$$-\int_0^h \frac{du}{\sqrt{u^{\frac{\sigma}{2}} + u^{2+\sigma}}} = -\int_0^h \frac{du}{u^{\frac{\sigma}{4}}\sqrt{1 + O(u^{2+\frac{\sigma}{2}})}} = -\frac{4}{4-\sigma} h^{1-\frac{\sigma}{4}} + o\left(h^{1-\frac{\sigma}{4}}\right),$$

respectively, as $h \to 0$. Moreover the third term in the right-hand side in (3.1) is calculated as

$$\begin{aligned}
&\int_h^\infty \left(\frac{1}{\sqrt{u^{\frac{\sigma}{2}} - h^{\frac{\sigma}{2}} + u^{2+\sigma} - h^{2+\sigma}}} - \frac{1}{\sqrt{u^{\frac{\sigma}{2}} + u^{2+\sigma}}} \right) du \\
&= \int_h^\infty \frac{h^{\frac{\sigma}{2}} + h^{2+\sigma}}{\sqrt{u^{\frac{\sigma}{2}} + u^{2+\sigma}}\sqrt{u^{\frac{\sigma}{2}} - h^{\frac{\sigma}{2}} + u^{2+\sigma} - h^{2+\sigma}}} \\
&\qquad \times \frac{1}{\sqrt{u^{\frac{\sigma}{2}} + u^{2+\sigma}} + \sqrt{u^{\frac{\sigma}{2}} - h^{\frac{\sigma}{2}} + u^{2+\sigma} - h^{2+\sigma}}} du
\end{aligned}$$

and, as $h \to 0$, if $0 < \sigma < \frac{4}{3}$ then the last term is

$$\int_0^\infty \frac{du}{(u^{\frac{\sigma}{2}} + u^{2+\sigma})^{\frac{3}{2}}} h^{\frac{\sigma}{2}} + o(h^{\frac{\sigma}{2}}) = \frac{2}{4+\sigma} B\left(\frac{4-3\sigma}{2(4+\sigma)}, \frac{4+3\sigma}{4+\sigma} \right) h^{\frac{\sigma}{2}} + o(h^{\frac{\sigma}{2}}),$$

and, if $\frac{4}{3} < \sigma < 2$ then the last term is

$$\begin{aligned}
\int_h^\infty \frac{du}{(u^{\frac{\sigma}{2}} + u^{2+\sigma})^{\frac{3}{2}}} h^{\frac{\sigma}{2}} + o(h^{\frac{\sigma}{2}}) &= \frac{4}{3\sigma-4}(h^{1-\frac{3}{4}\sigma} + o(h^{1-\frac{3}{4}\sigma}))h^{\frac{\sigma}{2}} + o(h^{\frac{\sigma}{2}}) \\
&= \frac{4}{3\sigma-4} h^{1-\frac{\sigma}{4}} + o\left(h^{1-\frac{\sigma}{4}}\right).
\end{aligned}$$

Hence the third term of (3.1) is evaluated as

$$\begin{aligned}
&\int_h^\infty \left(\frac{1}{\sqrt{u^{\frac{\sigma}{2}} - h^{\frac{\sigma}{2}} + u^{2+\sigma} - h^{2+\sigma}}} - \frac{1}{\sqrt{u^{\frac{\sigma}{2}} + u^{2+\sigma}}} \right) du \\
&= \begin{cases}
\frac{2}{4+\sigma} B\left(\frac{4-3\sigma}{2(4+\sigma)}, \frac{4+3\sigma}{4+\sigma} \right) h^{\frac{\sigma}{2}} + o\left(h^{\frac{\sigma}{2}}\right), & \text{if } 0 < \sigma < \frac{4}{3}, \\
-(\log h) h^{1-\frac{\sigma}{4}} + O\left(h^{1-\frac{\sigma}{4}}\right), & \text{if } \sigma = \frac{4}{3}, \\
\frac{4}{3\sigma-4} h^{1-\frac{\sigma}{4}} + o\left(h^{1-\frac{\sigma}{4}}\right), & \text{if } \frac{4}{3} < \sigma < 2,
\end{cases}
\end{aligned}$$

as $h \to 0$. Therefore the behavior of $T(h)$ as $h \to 0$ is given by
(3.2)

$$T(h) = \begin{cases}
\frac{2}{4+\sigma} B\left(\frac{\sigma}{4+\sigma}, \frac{4-\sigma}{8+2\sigma} \right) + \frac{2}{4+\sigma} B\left(\frac{4-3\sigma}{2(4+\sigma)}, \frac{4+3\sigma}{4+\sigma} \right) h^{\frac{\sigma}{2}} + o\left(h^{\frac{\sigma}{2}}\right), & \text{if } 0 < \sigma < \frac{4}{3}, \\
\frac{3}{8} B\left(\frac{1}{4}, \frac{1}{4}\right) - (\log h) h^{\frac{2}{3}} + O\left(h^{\frac{2}{3}}\right), & \text{if } \sigma = \frac{4}{3}, \\
\frac{2}{4+\sigma} B\left(\frac{\sigma}{4+\sigma}, \frac{4-\sigma}{8+2\sigma} \right) + \frac{16(2-\sigma)}{(3\sigma-4)(4-\sigma)} h^{1-\frac{\sigma}{4}} + o\left(h^{1-\frac{\sigma}{4}}\right), & \text{if } \frac{4}{3} < \sigma < 2.
\end{cases}$$

Since both $\frac{\sigma}{2}$ for $0 < \sigma < \frac{4}{3}$ and $1 - \frac{\sigma}{4}$ for $\frac{4}{3} < \sigma < 2$ in (3.2) are less than $\frac{2}{3}$, the function $T = \mathcal{B}f$ in this example is locally Hölder continuous with the exponent $\beta < \frac{2}{3}$. Note that each $\beta \in (0, \frac{2}{3})$ can be realized by some $\sigma \in (0, 2)$.

Though the function $T(h)$ is continuous even at $h = 0$ with $T(0) > 0$, the solution $f(u)$ is continued no longer to the left beyond $u = 0$. Because the function $T = \mathcal{B}f$ in this example is locally Hölder continuous with the exponent $\beta < \frac{2}{3}$ associated with σ, this example shows that the continuation of the solution f of $\mathcal{B}f = T$ is not guaranteed only by assuming that T is Hölder continuous with the Hölder exponent β in the case $\beta < \frac{2}{3}$.

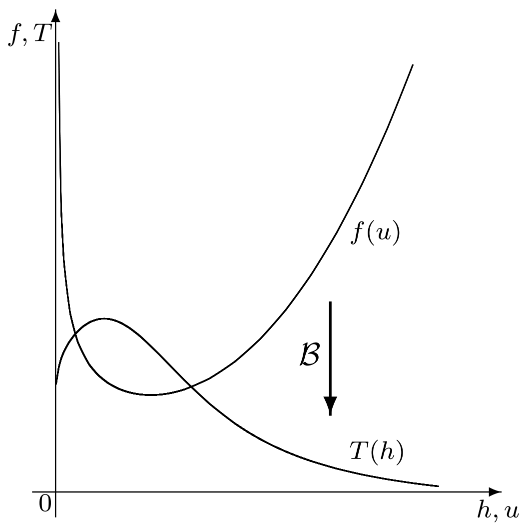

Figure 4. Example 3.7

§ 4. Proof keys

In this section, we give some basic ideas to prove results stated in previous sections. Some of inverse problems to determine nonlinear terms of differential equations from information of solutions to these equations are relevant to our inverse blow-up problem in an aspect of theme connected with nonlinear integral equations. Especially, (1) a classical inverse problem [1, 2, 4, 10, 12, 17, 20] (see also [11, Chapter 3] for an overview of the problem) to determine a restoring force in the Newtonian equation from the period function assigning a period to an amplitude including its generalized problem [16, 23] and (2) an inverse bifurcation problem [6, 7, 9, 19] to determine a nonlinear term in a class of nonlinear Sturm-Liouville equations are closely related with the inverse blow-up problem not merely in the aspect but also from a viewpoint of approaches to integral equations appearing in problems.

§ 4.1. Local discussion

To find a solution f of (1.2) in $[H,\infty)$, that is, to establish existence results of solutions to the equation, we require a method of a certain linearization. The method employed for the proof (for its details, refer to [13]) of Theorem 2.4 consists in a use of the implicit function theorem. The Fréchet derivative of (2.4) is computed, by reversing the order of integration, as

$$\mathcal{B}'(f_0)f(h) = -\frac{1}{2\sqrt{2}}\int_h^\infty f(r)dr\int_r^\infty \frac{du}{\left(\int_h^u f_0(\xi)d\xi\right)^{\frac{3}{2}}},$$

where f_0 is known, such as $f_0(u) = u^{1+\sigma}$. This linear operator is an integral operator of the first kind with a weak singularity at $r = h$. This singularity is the reason for a use of somewhat complicated Hölder spaces $\mathcal{C}^\alpha(I)_\eta$ in Theorem 2.4.

Though an analysis working with (2.1) directly leads us to a stronger existence result such as Theorem 2.5 than Corollary 2.6, in order to establish uniqueness results such as Theorems 2.8, 2.10 released from the Hölder spaces setting, it will be more convenient to work with a transformed form of equation (1.2), rather than work with it directly.

Let F be a primitive function of unknown function f. Because of $f > 0$, F becomes a monotonically increasing function. Hence we can define the inverse function $p(k)$ of F. Then $p(k)$ is a monotonically increasing function in some interval $[b,\infty)$. By viewing p as unknown function, equation (1.2) is recast as

$$\int_k^\infty \frac{p'(v)}{\sqrt{v-k}}dv = \sqrt{2}\,T(p(k)), \quad b \le k < \infty, \tag{4.1}$$

via the substitution $v = F(u)$ ($\Leftrightarrow u = p(v)$) and setting $k = F(h)$ ($\Leftrightarrow h = p(k)$). In this section that expands a local theory near $h = +\infty$, we can assume that h is large, and so, assume that k is large. Hence, in what follows, we let $b > 0$. Notice that if $p(k)$ is a solution of (4.1) then its parallel shifts $p(k+d)$ are also solution of the equation, which is passed on from 1-dimensional indefiniteness of indefinite integral F.

Equation (4.1) is a nonlinear Volterra equation, whose nonlinearity appears only in the right-hand side as a composition of a known (prescribed) function T and an unknown p. To rewrite (4.1) as an equation free from the derivative p', we act an operator $\int_k^\infty \frac{dr}{\sqrt{r-k}\,r}$ to the equation. Then, by an interchange of the order of integration and an elementary integral

$$\int_k^v \frac{dr}{\sqrt{r-k}\sqrt{v-r}\,r} = \frac{\pi}{\sqrt{vk}},$$

we obtain

$$\frac{1}{\sqrt{k}}\int_k^\infty \frac{p'(v)}{\sqrt{v}}dv = \frac{\sqrt{2}}{\pi}\int_k^\infty \frac{T(p(v))}{\sqrt{v-k}\,v}dv, \quad b \le k < \infty.$$

In addition, integrating the right-hand side by parts and then multiplying the resultant equation by k, we get

$$-p(k) + \frac{\sqrt{k}}{2} \int_k^\infty \frac{p(v)}{v^{\frac{3}{2}}} dv = \frac{\sqrt{2}}{\pi} k \int_k^\infty \frac{T(p(v))}{\sqrt{v-k}\, v} dv, \quad b \le k < \infty.$$

Consequently, by the substitution $v = kt$, we arrive at

$$p(k) - \frac{1}{2} \int_1^\infty \frac{p(kt)}{t^{\frac{3}{2}}} dt + \frac{\sqrt{2}}{\pi} \sqrt{k} \int_1^\infty \frac{T(p(kt))}{\sqrt{t-1}\; t} dt = 0, \quad b \le k < \infty. \tag{4.2}$$

Notice that the procedure from (4.1) to (4.2) is understood as a natural way through an inversion formula (see [8, page 171]) for the Erdélyi-Kober operator.

Unlike (1.2), the equation (4.2) is an integral equation of the second kind, although it still has a singularity at $t = 1$ in its third term. By virtue of that the principal part is the identity operator (see, e.g., [8, §4.7]), integral equations of the second kind are, in general, more amenable than those of the first kind, as is easily understood by remembering a solving method for a generalized Abel integral equation (see, e.g., [24, §41]). In light of this, we will work with (4.2).

To make our discussion clear, we focus our attention on an analysis around (2.15) with $c = 1$, namely, around

$$\mathcal{B} : f_0(u) = e^u \;\mapsto\; T_0(h) = \frac{\pi}{\sqrt{2}}\, e^{-\frac{h}{2}}.$$

Observing that the function $p(k)$ for $f_0(u) = e^u$ is given by $p_0(k) = \log k$ up to the parallel shift, we introduce a new, unknown function $\varphi(k) := p(k) - \log k$. Then, as is easily verified by a elementary computation, (4.2) is rewritten as an equation

$$\mathcal{F}(\varphi, T)(k) = 0, \quad b \le k < \infty,$$

for unknown φ, where

$$\mathcal{F}(\varphi, T)(k) := \varphi(k) - \frac{1}{2} \int_1^\infty \frac{\varphi(kt)}{t^{\frac{3}{2}}} dt + \frac{\sqrt{2}}{\pi} \sqrt{k} \int_1^\infty \frac{T(\varphi(kt) + \log(kt))}{\sqrt{t-1}\; t} dt - 2.$$

It turns out that the pair of function spaces

$$X := \{\varphi \in C[b, \infty) \;:\; ||\varphi||_X := \sup_{b \le k < \infty} |\varphi(k)| < \infty\},$$
$$Y := \{T \in C^1[a, \infty) \;:\; \lim_{h \to \infty} T(h) = 0,\; ||T||_Y := \sup_{a \le h < \infty} |e^{\frac{h}{2}} T'(h)| < \infty\},$$

works well for the map $\mathcal{F}$, reflecting the principal part of $\mathcal{F}$ to be the identity. Actually, under the assumption $a < \log b$ with $b > 0$, the transform $\mathcal{F}$ becomes a C^1-map of an open neighborhood of $(0, T_0)$ in $X \times Y$ to X, whose Fréchet derivative $\mathcal{F}_\varphi(0, T_0)$ of $\mathcal{F}$ in φ at $(0, T_0)$ is computed as

$$\mathcal{F}_\varphi(0, T_0)\varphi(k) = \varphi(k) - \frac{1}{2}\int_1^\infty \frac{\varphi(kt)}{t^{\frac{3}{2}}}dt - \frac{1}{2}\int_1^\infty \frac{\varphi(kt)}{\sqrt{t-1}\, t^{\frac{3}{2}}}dt.$$

Thus we encounter a linear, integral operator (the right-hand side of this formula) of the second kind. Unfortunately it is not an integral operator for which the method of successive approximations works well. However, fortunately, it is a Fredholm operator with a positive index. To explain it we introduce the notation

$$J_\Phi \varphi(k) = \int_1^\infty \Phi(t)\varphi(kt)dt, \quad b \le k < \infty. \tag{4.3}$$

Then the Fréchet derivative $\mathcal{F}_\varphi(0, T_0)$ is represented in the form $I - J_\Phi$ in terms of the operator $J_\Phi : X \to X$. The concrete form of Φ is given by

$$\Phi(t) = \frac{1}{2\, t^{\frac{3}{2}}}\left(1 + \frac{1}{\sqrt{t-1}}\right), \quad 1 < t < \infty. \tag{4.4}$$

If we set $k = be^x$, $t = e^{y-x}$, $\psi(x) = \varphi(be^x)$, and $\ell(x) = e^{-x}\Phi(e^{-x})\chi_{(-\infty,0]}(x)$, where $\chi_{(-\infty,0]}$ is the characteristic function of the interval $(-\infty, 0]$, then (4.3) can be rewritten as

$$L\psi(x) = \int_0^\infty l(x-y)\psi(y)dy, \quad 0 \le x < \infty.$$

The right-hand side is what we know as a Wiener-Hopf integral operator (see [5, 8, 15] for the classical theory of the operator). Accordingly the Fredholm-ness of $I - J_\Phi$ is passed on from that of the Wiener-Hopf operator $I - L$. That is, in non-resonant cases, its kernel $\operatorname{Ker}(I - J_\Phi)$ is controlled by zeros of a complex function (called the symbol)

$$D_\Phi(z) := \int_1^\infty \Phi(t)\, t^{-z} dt, \quad \operatorname{Re} z \ge 0.$$

Thus, as a direct rewriting of known results (see, e.g., [8, §4.4], [15, §9]) on the standard Wiener-Hopf operator $I - L$ through the setting connecting J_Φ with L, we obtain:

Lemma 4.1 (Fredholm-ness). *Let $\Phi \in L^1(1, \infty)$. If $D_\Phi(z) \ne 0$ on the imaginary axis, then:*

(1) *$I - J_\Phi$ is a surjective, bounded linear operator of X onto X.*

(2) $\operatorname{Ker}(I - J_\Phi)$ *is a finite dimensional subspace in X, whose dimension N equals the number of zeros of $1 - D_\Phi(z)$ in the right half-plane* $\operatorname{Re} z > 0$*, counted with multiplicities.*

It follows from the Riemann-Lebesgue theorem that $1 - D_\Phi(z) \to 1$ as $|z| \to \infty$ in $\operatorname{Re} z \geq 0$. Hence, assertion (2) in the lemma is a consequence from the unicity theorem in theory of complex functions. Moreover, by the argument principle in the theory, N can be calculated through the so-called index of the Wiener-Hopf operator:

$$\operatorname{ind}(I - J_\Phi) = \frac{1}{2\pi} \int_{i\,\infty}^{-i\,\infty} d_\xi \arg\left(1 - D_\Phi(\xi)\right).$$

The right-hand side indicates how many times a curve $1 - D_\Phi(\xi)$ winds around the origin when ξ moves from $+i\,\infty$ to $-i\,\infty$. It should also be noted that if z is a zero of $1 - D_\Phi(z)$ then $\varphi(k) = k^{-z} \in X$ is a solution of equation $(I - J_\Phi)\varphi = 0$, which is verified by the direct substitution of k^{-z} into the equation.

For our Φ in (4.4), one can show that the curve moves such as Figure 5: the curve $1 - D_\Phi(iy)$ starting from 1 as $y = +i\infty$ moves in the upper half-plane when $y > 0$, passes through the point -1 when $y = 0$, and then returns to 1 as $y = -i\infty$ through the lower half-plane when $y < 0$.

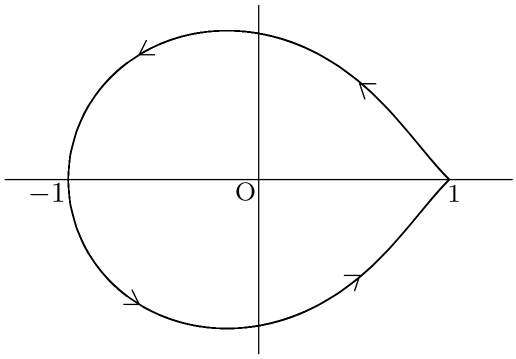

Figure 5. Curve $1 - D_\Phi(\xi)$ for (4.4).

Thus $\dim \operatorname{Ker}(I - J_\Phi) = 1$ for Φ in (4.4). This is just corresponding to that the parallel shifts $p(k+d)$ are also solutions of (4.2), reflecting a 1-dimensional indefiniteness of indefinite integral; the parallel shift $p(k+d)$ satisfies (4.2) and so, the canonical shift $\varphi(k+d) + \log(1 + \frac{k}{d})$ induced from

$$\begin{tikzcd} p(k) \arrow[r, "\text{parallel shift}"] & p(k+d) \\ \varphi(k) \arrow[u, "+\log k"] \arrow[r, "\text{canonical shift}"] & \varphi(k+d) + \log(1 + \frac{d}{k}) \arrow[u, "+\log k"'] \end{tikzcd}$$

satisfies $\mathcal{F}(\varphi, T) = 0$. Moreover, the zero of $D_\Psi(z)$ in $\operatorname{Re} z > 0$ is $z = 1$, as is seen by the direct computation $1 - D_\Phi(1) = 0$. This is the same as saying that $(I - J_\Phi)\frac{1}{k} = 0$.

In light of this observation, we decompose X as a direct sum of the subspace $X_1 := \mathrm{span}\left\{\frac{1}{k}\right\}$ and a complementary closed subspace X_2 to have $X = X_1 \oplus X_2$, and denote the corresponding (unique) decomposition by $\varphi = \varphi_1 + \varphi_2$, where $\varphi \in X$, $\varphi_i \in X_i$. By setting $\phi_0(k) := \frac{1}{k}$, X_1 is expressed as $X_1 = \{\lambda\phi_0 : \lambda \in \mathbb{R}\}$, and in addition, an open neighborhood U_1 of 0 in X_1 is expreseed as $U_1 = \{\lambda\phi_0 : |\lambda| \text{ is small}\}$.

We now let $\tilde{\mathcal{F}}$ be the C^1-map of an open neighborhood of $((0, T_0), 0)$ in $(X_1 \times Y) \times X_2$ defined by $\tilde{\mathcal{F}}((\varphi_1, T), \varphi_2) = \mathcal{F}(\varphi_1 + \varphi_2, T)$. Then $\tilde{\mathcal{F}}((0, T_0), 0) = \mathcal{F}(0, T_0) = 0$ with the Fréchet derivative $\tilde{\mathcal{F}}_{\varphi_2}((0, T_0), 0) = \mathcal{F}_\varphi(0, T_0)|_{X_2} = (I - J_\Phi)|_{X_2}$, which is an isomorphism of X_2 onto X. Therefore, by the implicit function theorem (see, e.g., [3, Theorem 2.3]), we arrive at:

Proposition 4.2. *There exist open neighborhoods U_1 of 0 in X_1, V of T_0 in Y, U_2 of 0 in X_2, and a C^1-map $\mathcal{A} : U_1 \times V \to U_2$ with $\mathcal{A}(0, T_0) = 0$ such that*

$$\mathcal{F}(\varphi, T) = 0, \;\; \varphi \in U_1 \times U_2, \;\; T \in V \iff \varphi(k) = \lambda\phi_0 + \mathcal{A}(\lambda\phi_0, T) \;\text{ with small } |\lambda|.$$

This is the key proposition for the proof of Theorem 2.10. The proof is completed after careful checks of the facts: (1) a trajectory (see Figure 6) of solutions $\varphi = \lambda\phi_0 + \mathcal{A}(\lambda\phi_0, T)$ with $\varphi_T := \mathcal{A}(0, T)$ is filled with these canonical shifts near $(0, T_0)$, (2) scaling operators $f(u) \mapsto e^{-R} f(u + R)$, $T(h) \mapsto e^{\frac{R}{2}} T(h + R)$ do work well, instead of (2.1), in the framework of this function spaces setting.

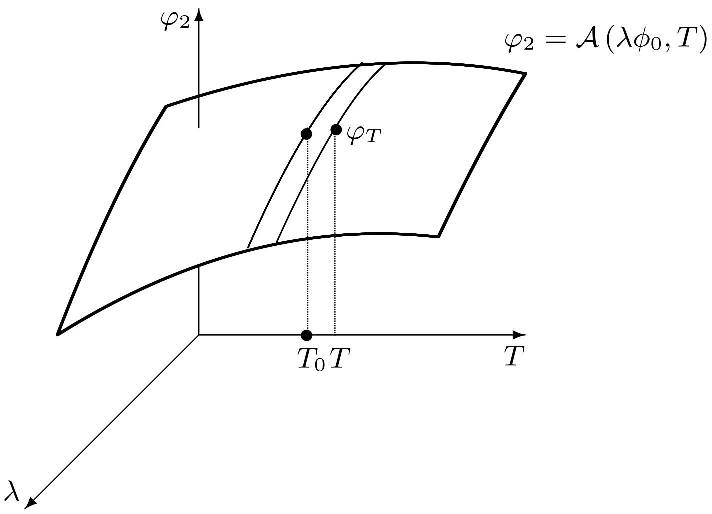

Figure 6. Map $\mathcal{A}(\lambda\phi_0, T)$.

For a development of local theory to the inverse blow-up problem, we have employed the implicit function theorem with a decomposition of a function space and the Fredholm-ness of a Wiener-Hopf operator of the second kind. A similar use of the theorem and Fredholm-ness in a somewhat different operator of the Wiener-Hopf type can be found in [7].

§ 4.2. Global discussion

A basic, global feature of the inverse blow-up problem is given in Theorem 3.1. Actually the proof of the theorem is heavily based on an observation in [17] for a study of the classical inverse problem to a restoring force in the Newtonian equation from the period function. After the observation, this problem has been studied by many authors [1, 2, 4, 20] in the framework of local theory, or of real analytic functions. A complete, global answer to the problem was given by [10, 12] in relatively recent years, which is successfully extended by [23] to the inverse problem for a quasilinear differential equation. Our strategy to prove Theorem 3.5 is along the approach in [1, 10, 23].

The first step of the strategy is also a use of (4.1). However, in turn, we suppose that the section of $p(k)$ in the right, say on $[0,\infty)$, is viewed as known. So we rewrite (4.1) as

$$\int_k^0 \frac{p'(v)}{\sqrt{v-k}}dv - \sqrt{2}\,T(p(k)) = -\int_0^\infty \frac{p'(v)}{\sqrt{v-k}}dv, \quad b \le k \le 0. \tag{4.5}$$

Here the right-hand side is known, and the objective is to obtain a solution of this equation on the interval $[b,0]$. Notice that, for a nonlinear term f to be defined as a positive, continuous function, it is necessary that $p(k)$ belongs to $C^1[b,0]$ with $p'(k)>0$ on $[b,0]$, remembering f is the derivative of the inverse function of p.

We now let

$$(I^\delta \phi)(k) = \frac{1}{\Gamma(\delta)}\int_k^0 \frac{\phi(r)}{(r-k)^{1-\delta}}dr, \quad \delta > 0,$$

where Γ is the Gamma function. This operator is a standard Riemann-Liouville integral operator (see, e.g., [18, Eq.(2.18)]), which has the semigroup property $I^{\delta_1}I^{\delta_2} = I^{\delta_1+\delta_2}$ on $L^1(b,0)$ for $\delta_1,\delta_2>0$. In particular, $I^{\frac12}I^{\frac12} = I$ is a regular integration from k to 0.

The first term of equation (4.5) is written as $I^{\frac12}\sqrt{\pi}\,p'$. Hence, by applying $I^{\frac12}$ to the both sides, we deduce

$$p(k) + \frac{\sqrt{2}}{\pi}\int_k^0 \frac{T(p(r))}{\sqrt{r-k}}dr = q(k), \quad b \le k \le 0. \tag{4.6}$$

where

$$q(k) := p(0) + \frac{1}{\sqrt{\pi}} I^{\frac12}\int_0^\infty \frac{p'(v)}{\sqrt{v-k}}dv,$$

is a known function. The key proposition for the proof of Theorem 3.5 is:

Proposition 4.3. *Suppose that T is a positive, locally Lipschitz continuous function and let $p(k)$ be a continuous solution of* (4.6). *Then:*

(1) *If $p(k)$ is continuously differentiable at $k=0$ then $p(k)$ belongs to $C^1[b,0]$.*

(2) *If $p'(0)>0$ then $p'(k)>0$ on $[b,0]$.*

Even though the character of nonlinear integral equation (4.6) in assertion (1) of this proposition is naturally understood (by a similar observation to in [1]) as a reflection of a qualitative property (the smoothing property) of the Riemann-Liouville integral operator, the character in the assertion (2) is somewhat remarkable; at a first glance, it can not be expected that the global monotonicity of a solution p is passed on from its local monotonicity near $k = 0$. In what follows, we overview the proof of the assertion, leaving its details to [14].

We now employ a Riemann-Liouville differential operator

$$D^{\delta} = DI^{1-\delta},$$

where $D = -\frac{d}{dk}$ (see, e.g., [18, Eq.(2.23)]). Notice that, for $\phi \in C^{\nu+\delta}[b,0)_{\eta+\delta}$, the following Weyl-Marchaud formula holds:

$$D^{\delta}\phi(k) = \frac{1}{\Gamma(1-\delta)}\left(\frac{\phi(k)}{(-k)^{\delta}} - \delta\int_k^0 \frac{\phi(r)-\phi(k)}{(r-k)^{\delta+1}}dr\right), \quad b \le k < 0,$$

provided that $\eta > -1$, $0 < \nu < \nu + \delta \le 1$. By use of this differential operator, (4.6) leads to

$$D^{\frac{1}{2}-\varepsilon}p' + D^{1-\varepsilon}q_0 = \sqrt{\tfrac{2}{\pi}}\,D^{1-\varepsilon}(T \circ p), \tag{4.7}$$

where ε is any small, positive number and

$$q_0(k) := \frac{1}{\sqrt{\pi}}\int_0^{\infty} \frac{p'(v)}{\sqrt{v-k}}dv.$$

Assertion (2) is proved by a contradiction argument; assume $p'(k)$ were negative at some $k \in [b,0)$ and let $\kappa < 0$ be the first point of k at which $p'(k) = 0$. Then, in view of $p'(k) > 0$ on $(\kappa, 0]$, $p'(\kappa) = 0$, we find that $(D^{\frac{1}{2}-\varepsilon}p')(\kappa)$ is negative uniformly for small, positive ε, with the aid of the Weyl-Marchaud formula above, and, in view of monotonic increase of $q_0(k)$ on $[b,0)$, we verify that $D^{1-\varepsilon}q_0(\kappa)$ becomes sufficiently small as ε becomes small. Accordingly, the value at κ of the left-hand side in (4.7) is negative for sufficiently small ε. On the other hand, in view of the Lipschitz continuity of T, we show that the value at κ of the right-hand side in (4.7) tends to 0 as $\varepsilon \to 0$, also with the aid of the Weyl-Marchaud formula. This causes the contradiction,

This contradiction argument is considered as a modification, with a use of a fractional derivative, of a standard discussion in order to show a function that is positive at a point remains positive ever in the left of the point. It is expected that not merely a qualitative use of the so-called fractional calculus but also its quantitative use will give an effective way to study other inverse problems arising from mathematical sciences.

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